Lecture 37

- Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

$$dA = r\,dr\,d\theta$$

- A lamina, D, with density function $\rho(x,y)$:

  i) mass $= m = \iint_D \rho\, dA$

  ii) moment about x-axis $= M_x = \iint_D y\rho\, dA$

  iii) moment about y-axis $= M_y = \iint_D x\rho\, dA$.

  iv) center of mass $= (\bar{x}, \bar{y})$ where $\bar{x} = \dfrac{M_y}{m}$, $\bar{y} = \dfrac{M_x}{m}$

- A solid, E, with density function $\rho(x,y,z)$:

  i) mass $= m = \iiint_E \rho\, dV$

  ii) moment about:
  - yz-plane $= M_{yz} = \iiint_E x\rho\, dV$
  - xz-plane $= M_{xz} = \iiint_E y\rho\, dV$
  - xy-plane $= M_{xy} = \iiint_E z\rho\, dV$

  iii) center of mass $= (\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{M_{yz}}{m},\quad \bar{y} = \frac{M_{xz}}{m},\quad \bar{z} = \frac{M_{xy}}{m}.$$

- Be able to read the region of integration from the bounds of a triple integral (usually in order to change the order of integration).

e.g.: $\int_0^1 \int_x^{2x} \int_0^y 2xyz \, dz \, dy \, dx$ is an integral over the region bounded by the planes:

$z=0$, $z=y$, $y=x$, $y=2x$, $x=0$, and $x=1$.

- Cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$dV = r \, dr \, d\theta \, dz$$

- Spherical coordinates: $x = \rho\cos\theta\sin\varphi$, $y = \rho\sin\theta\sin\varphi$, $z = \rho\cos\varphi$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

- General Change of Variables
  - Used to either simplify the region or the integral
  - For a transformation $T(u,v) = (x(u,v), y(u,v))$ (i.e. $x = x(u,v)$, $y = y(u,v)$) which sends a region S in the uv-plane to a region R in the xy-plane (i.e. $T(S) = R$),

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

- $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- You can use the fact that $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\left(\dfrac{\partial(u,v)}{\partial(x,y)}\right)}$ if when you choose u & v, x & y are hard to solve for.

- If C is parametrized by $\vec{r}(t)$, $a \le t \le b$

i) $\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

ii) $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\int_C P\,dx + Q\,dy = \int_0^{2\pi} \left(P(\vec{r}(t))\,x'(t) + Q(\vec{r}(t))\,y'(t) \right) dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_C \vec{F} \cdot d\vec{r}, \quad \vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

iii) $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$, $\vec{F}(x,y,z) = \langle P, Q, R \rangle$

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left(P(\vec{r}(t))\,x'(t) + Q(\vec{r}(t))\,y'(t) + R(\vec{r}(t))\,z'(t) \right) dt$$

- $$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

- $$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$: Fundamental Theorem of Line Integrals

C is a path starting at A and ending at B.

- A vector field, $\vec{F}$, is conservative if
$$\vec{F} = \nabla f$$
for some scalar function f. f is called a potential function for $\vec{F}$.

- To try to find a potential, f, for $\vec{F} = \langle P, Q \rangle$, we assume $\vec{F} = \nabla f = \langle f_x, f_y \rangle$, use $P = f_x$ & $Q = f_y$, and integrate to try to find f. Compare with the equation you didn't integrate to solve for the arbitrary function obtained from the integration.

- Green's Theorem
$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$
where $C = \partial D$.